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# On the stability of forward and rearward slug flows due to the motion of a body through a perfectly conducting liquid

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The stability of forward and rearward slug flows due to the motion of a body through a perfectly conducting liquid with an embedded magnetic field aligned to the direction of motion is studied. It is found that, for sub-Alfvénic flows, the forward and rearward slug flows predicted by Stewartson, Leibovich and Ludford are stable. For super-Alfvénic flows, if at all meaningful, both the forward and the rearward slug flows are stable also.

## 1. Introduction

In an early paper in the rapidly developing field of magnetohydrodynamics, Sears & Resler (1959) considered the problem of steady thin airfoil theory in an infinitely conducting liquid; for the case of an embedded magnetic field aligned with the undisturbed flow direction, it was concluded that the flow was irrotational and the same as for the non-conducting case. The same problem was later considered by Stewartson (1960), and by Ludford & Leibovich (1965) for the case of motion slow compared with the Alfvén speed. In both papers it was concluded that the flow differed radically from the non-conducting case. Their restriction to slow flow was removed by Leibovich & Ludford (1965). Stewartson, Leibovich, and Ludford found steady flow patterns characterized by infinite slabs of fluid, included between two planes parallel to the applied magnetic field, which are carried along by the body as if solid. Outside the slabs of dead fluid, conditions are essentially undisturbed. The planes bounding the slugs are surfaces of discontinuity of the tangential components of velocity and magnetic field, and are thus vortex and current sheets.

Since slug flows in ordinary hydrodynamics are unstable, it is of interest to examine the stability of the hydromagnetic slug flows. The stability of single combined vortex and current sheets has been discussed by Syrovatskii (1953), and Michael (1955), who showed that the magnetic field reduces the classical instability of a vortex sheet. This conclusion is unchanged by the presence of a second, parallel sheet. In fact, it is found that all slug flows created by the motion of thin airfoils, as discussed above, are stable.

## 2. Analysis

Consider a two-dimensional inviscid slug flow with a slug width $2l$. Thus, there are two parallel current–vortex sheets at $y = l$ and $y = -l$. Across the sheets, the magnetic and the velocity fields are discontinuous both in magnitude and orientation, but the field vectors are parallel to the planes of the sheets. Such a discontinuous flow is compatible with the governing equations (cf. Landau & Lifshitz 1960, pp. 224–225).

For an incompressible, perfectly conducting, and inviscid fluid, the relevant linearized disturbance equations are given by (Syrovatskii 1953)

$$\operatorname{div} \mathbf{U}' = 0, \quad \operatorname{div} \mathbf{V}' = 0, \tag{1}$$

$$\partial \mathbf{U}'/\partial t = (\mathbf{U}\,.\operatorname{grad})\,\mathbf{V}' - (\mathbf{V}\,.\operatorname{grad})\,\mathbf{U}', \tag{2}$$

$$\partial \mathbf{V}'/\partial t + (\mathbf{V}\,.\operatorname{grad})\,\mathbf{V}' = -(1/\rho)\operatorname{grad}(P' + \rho\mathbf{U}\,.\,\mathbf{U}') + (\mathbf{U}\,.\operatorname{grad})\,\mathbf{U}', \tag{3}$$

where $\mathbf{U} = \mathbf{H}/(4\pi\rho)^{\frac{1}{2}}$ is the Alfvén velocity, $\mathbf{V}$ the constant flow velocity, $\rho$ the density of the fluid, $\mathbf{H}$ the magnetic field intensity, and $P$ the dynamic pressure. Primed (accented) quantities are disturbance ones. From (1) and (3), one obtains

$$\Delta(P' + \rho\mathbf{U}\,.\,\mathbf{U}') = 0, \tag{4}$$

where $\Delta$ is the Laplacian operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Assume that the disturbances have the form

$$\{\mathbf{U}', \mathbf{V}', P'\} = \{\mathbf{u}(y), \mathbf{v}(y), p(y)\} \exp\left[i(\mathbf{k}\,.\,\mathbf{r} - \omega t)\right], \tag{5}$$

where $\mathbf{k}$ is in the $(x, z)$-plane. By eliminating $v_y$ from the $y$-components of equations (2) and (3), one has

$$i\left[\frac{(\mathbf{k}\,.\,\mathbf{V} - \omega)^2}{\mathbf{k}\,.\,\mathbf{U}} - \mathbf{k}\,.\,\mathbf{U}\right] u_y = -\frac{1}{\rho}\frac{d}{dy}(p + \rho\mathbf{U}\,.\,\mathbf{u}). \tag{6}$$

The general solution for equation (4) is given by

$$p_j + \rho_j \mathbf{U}_j\,.\,\mathbf{u}_j = A_j e^{ky} + B_j e^{-ky}, \tag{7}$$

where $j = 1, 2, 3$ depending upon whether $y > l$, $|y| < l$, or $y < -l$, $k = (\mathbf{k}\,.\,\mathbf{k})^{\frac{1}{2}}$, and $A_j$, $B_j$ are integration constants.

The appropriate boundary conditions at an interface separating two regions are shown by Landau & Lifshitz (1960) to be the continuity of total pressure, and the vanishing of the normal magnetic and velocity fields relative to the interface. Letting the displacement of a current–vortex sheet be

$$\delta = \zeta \exp\left[i(\mathbf{k}\,.\,\mathbf{r} - \omega t)\right], \tag{8}$$

where $\zeta$ is a constant, the matching conditions are

$$p_j + \rho_j \mathbf{U}_j\,.\,\mathbf{u}_j = p_{j+1} + \rho_{j+1}\mathbf{U}_{j+1}\,.\,\mathbf{u}_{j+1}, \tag{9}$$

$$u_{y_j} - i(\mathbf{k}\,.\,\mathbf{U}_j)\,\zeta = 0, \tag{10}$$

$$u_{y_{j+1}} - i(\mathbf{k}\,.\,\mathbf{U}_{j+1})\,\zeta = 0, \tag{11}$$

where $j = 0$ or $1$. Eliminating $\zeta$ from (10) and (11), we have

$$u_{y_j}/\mathbf{k}\,.\,\mathbf{U}_j = u_{y_{j+1}}/\mathbf{k}\,.\,\mathbf{U}_{j+1}. \tag{12}$$

By substituting for $p_j + \rho_j \mathbf{U}_j . \mathbf{u}_j$ from (7) into (6), (9) and (12), and remembering that the base flow conditions in the regions $j = 1, 3$ are identical, the eigenvalue equation for the disturbance is given by

$$\frac{(\mathbf{k}.\mathbf{V}_1 - \omega)^2 - (\mathbf{k}.\mathbf{U}_1)^2}{(\mathbf{k}.\mathbf{V}_2 - \omega)^2 - (\mathbf{k}.\mathbf{U}_2)^2} = -\frac{\rho_2}{\rho_1} K(kl), \tag{13}$$

where $K(kl)$ is either $\coth(kl)$ or $\tanh(kl)$, depending on the symmetry or antisymmetry of the disturbance.

Equation (13) is of the second degree in $\omega$. Therefore, the condition that $\omega$ has complex roots gives the condition for instability, which is

$$(1+S)\,[(\mathbf{k}.\mathbf{U}_1)^2 + S(\mathbf{k}.\mathbf{U}_2)^2] < S[\mathbf{k}.(\mathbf{V}_1 - \mathbf{V}_2)]^2, \tag{14}$$

where $S = (\rho_2/\rho_1)\,K(kl)$.

When there is no magnetic field, the condition (14) is automatically satisfied; therefore, the flow is unstable. It is clear that magnetic fields tend to stabilize the flow, except when the disturbance-wave propagation direction is perpendicular to the magnetic fields.

In the case corresponding to the flow described by Leibovich & Ludford (1965), the flow field and the magnetic field are in the same direction. There is no fluid motion inside the sheets, so $\mathbf{V}_2 = 0$. Let

$$m = |\mathbf{V}_1|/|\mathbf{U}_1| \tag{15}$$

be the Alfvén number. The Alfvén speed between the sheets is then

$$\mathbf{U}_2 = (1 \pm m)\,\mathbf{U}_1 \tag{16}$$

in the flow of interest, and so the condition for instability takes the form

$$[S(1 \pm m)]^2 + 2S(1 \pm m) + 1 < 0. \tag{17}$$

Since the quadratic is a perfect square, the slugs engendered by airfoil motion are stable.

For sub-Alfvénic flows, the forward and rearward slug flows predicted by Stewartson, and Leibovich & Ludford are stable. Super-Alfvénic slug flows, if at all meaningful, are also stable.

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